WAVE COMPUTATION ON THE HYPERBOLIC DOUBLE DOUGHNUT

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Abstract. We compute the waves propagating on the compact surface of constant negative curvature and genus 2. We adopt a variational approach using finite elements. We have to implement the action of the fuchsian group by suitable boundary conditions of periodic type. A spectral analysis of the wave allows to compute the spectrum and the eigenfunctions of the Laplace-Beltrami operator. We test the exponential decay due to a localized dumping and the ergodicity of the geodesic flow.

I. Introduction.

The Hyperbolic Double Doughnut $\mathbf{K}$ is the compact manifold of negative constant curvature with two holes. We can define it by the quotient of the hyperbolic Poincaré disc $\mathbf{D}$, by some Fuchsian group Γ. Alternatively, we can construct it as the quotient of the so-called Dirichlet polygon, or fundamental domain $\mathcal{F} \subset \mathbf{D}$ by a suitable relation of equivalence $\sim$:

$$\mathbf{K} = \mathbf{D}/\Gamma = \mathcal{F}/\sim .$$

This beautiful object has many fascinating properties as regards the classical and quantum chaos (classical references are [1], [5]). Several important computational investigations of the spectrum were performed by using a stationnary method by R. Aurich and F. Steiner [4]. Moreover there has been much recent interest for the cosmological models with non trivial topology (a seminal work is the famous "Cosmic Topology" by M. Lachièze-Rey and J-P. Luminet [7]). In this context, $\mathbf{K}$ has been studied as a paradigm in [3], where a schema based on the finite differences on a euclidean grid was used to solve the D'Alembertian. In this paper we compute the solutions of the wave equations in the time domain, by using a variational method and a discretization with finite elements on very fine meshes. The domain of calculus is the Dirichlet polygon, therefore the initial Cauchy problem on the manifold without booundary $\mathbf{K}$, becomes a mixed problem on $\mathcal{F}$ and the action of the Fuchsian group is expressed as boundary conditions on $\partial\mathcal{F}$, analogous to periodic conditions. These boundary constraints are implemented in the choice of the basis of finite elements. By this way we obtain very accurate results on the transient waves. We test these results by performing a Fourier analysis of the transients waves that allows to find the first eigenvalues of the Laplace-Beltrami operator $\Delta_{\mathbf{K}}$ on $\mathbf{K}$. We compute also the solutions of the damped wave equation

$$\partial_t^2\psi - \Delta_{\mathbf{K}}\psi + a\partial_t\psi = 0.$$

When $0 \leq a \in L^\infty(\mathbf{K})$ and $a > 0$ on $\partial\mathcal{F}$, the ergodicity of the geodesic flow assures that the geometric control condition of Rauch and Taylor [9] is satisfied. Our numerical experiments agree with their theoretical results, stating that the energy decays exponentially.

II. The Hyperbolic Double Doughnut.

In this part we describe the construction of the Hyperbolic Double Doughnut. First we recall some important properties of the 2-dimensional hyperbolic geometry. It is convenient to use the representation of the hyperbolic space by using the Poincaré disc

$$\mathbf{D} := \left\{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 < 1\right\}, \tag{II.1}$$

endowed with the metric expressed with the polar coordinates by

$$ds^2_{\mathbf{D}} = \frac{4}{(1-r^2)^2}\mathrm{d}r^2 + 4\frac{r^2}{(1-r^2)^2}\mathrm{d}\varphi^2 = \frac{4}{(1-x^2-y^2)^2}[\mathrm{d}x^2+\mathrm{d}y^2]. \tag{II.2}$$

It is useful to use the complex parametrization $z = x+iy$. We have to carefully distinguish the euclidean distance

$$d(z,z') = \mid z - z' \mid, \tag{II.3}$$

and the hyperbolic distance associated with the hyperbolic metric, given by

$$\cosh d_H(z,z') = 1 + \frac{2|z-z'|^2}{(1-|z|^2)(1-|z'|^2)}. \tag{II.4}$$

We remark that

$$d(0,z) = \tanh\frac{d_H(0,z)}{2}$$

hence the euclidean circles centerd in 0 are hyperbolic circles, and more generally, all the hyperbolic circles $\{z';\ \ d_H(z',z_0)=R\}$, with $R>0$, $z_0\in\mathbf{D}$, are euclidean circles. The invariant measure $d\mu_H$ on the Poincaré disc allows to compute the area of any Lebesgue measurable subset $X\subset\mathbf{D}$ by the formula

$$\mu_H(X) = \int_X \frac{4}{(1-|z|^2)^2}\, dx\, dy, \tag{II.5}$$

in particular the hyperbolic area of a disc $D_H(0,R) := \{z;\ d_H(z,0)\le R\}$ is :

$$\mu_H(D_H(0,R)) = 4\pi\sinh^2\left(\frac{R}{2}\right). \tag{II.6}$$

The group of the isometries of $\mathbf{D}$ is generated by three kinds of transformations.

(1) The Rotations of angle $\varphi_0\in\mathbb{R}$

$$R_{\varphi_0}(z) = e^{i\varphi_0}z,$$

and so R_{φ_0} is defined in the (x,y)-coordinates by the matrix

$$R_{\varphi_0} = \begin{pmatrix} e^{i\frac{\varphi_0}{2}} & 0 \\ 0 & e^{-i\frac{\varphi_0}{2}} \end{pmatrix}.$$

(2) The Boosts, or Möbius transforms, associated with $\tau_0\in\mathbb{R}$:

$$T_{\tau_0}(z) = \frac{\cosh\frac{\tau_0}{2}z + \sinh\frac{\tau_0}{2}}{\sinh\frac{\tau_0}{2}z+\cosh\frac{\tau_0}{2}}$$

expressed in (x,y)-coordinates by the matrix

$$T_{\tau_0} = \begin{pmatrix} \cosh\frac{\tau_0}{2} & \sinh\frac{\tau_0}{2} \\ \sinh\frac{\tau_0}{2} & \cosh\frac{\tau_0}{2} \end{pmatrix}.$$

We remark that

$$\cosh d_H(z,T_{\tau_0}(z)) = 1 + 2\frac{|z^2-1|^2}{(1-|z|^2)^2}\sinh^2\left(\frac{\tau_0}{2}\right),$$

therefore

$$\forall z\in]-1,1[,\ \ d_H(z,T_{\tau_0}(z)) = \tau_0. \tag{II.7}$$

(3) The Symmetry

$$S(z) = \overline{z}.$$

Finally we recall that the geodesics of the Poincaré disc are the diameters and all the arcs of circles that intersect orthogonally the boundary of the disc.

Now we are ready to describe the double doughnut that is the quotient of the hyperbolic plane by the Fuchsian group of isometries, Γ, generated by the four transforms g_0, g_1, g_2, g_3, where

$$g_k \;=\; R_{k\frac{\pi}{4}}\, T_{\tau_1}\, R_{-k\frac{\pi}{4}} \tag{II.8}$$

with

$$\tanh\frac{\tau_1}{2} = \sqrt{\sqrt{2}-1}. \tag{II.9}$$

The matrix of g_k is given by :

$$g_k = \begin{pmatrix} 1+\sqrt{2} & \sqrt{2+2\sqrt{2}}\, e^{ik\frac{\pi}{4}} \\ \sqrt{2+2\sqrt{2}}e^{-ik\frac{\pi}{4}} & 1+\sqrt{2} \end{pmatrix}. \tag{II.10}$$

These isometries g_k satisfy the relation :

$$(g_0 g_1^{-1} g_2 g_3^{-1})(g_0^{-1} g_1 g_2^{-1} g_3) \;=\; I_d. \tag{II.11}$$

The *Hyperbolic Double Doughnut* is the quotient manifold

$$\mathbf{K} := \mathbf{D}/\Gamma, \tag{II.12}$$

endowed with the hyperbolic metric $ds^2_{\mathbf{K}}$ induced by $ds^2_{\mathbf{D}}$. We know, see e.g. [1], [2], [5], that $\mathbf{K}$ is a two dimensional C^∞ compact manifold, without boundary, its sectional curvature is constant, equal to -1, and its genus, that is the number of "holes", is 2. The geodesic flow is very chaotic : it is ergodic, mixing (theorems by G.Hedlung, E. Hopf), Anosov and Bernouillian (D. Ornstein, B. Weiss).

To perform the computations of the waves on the doughnut, it is very useful to represent it by a minimal subset $\mathcal{F} \subset \mathbf{D}$ and a relation of equivalence $\sim$ such that

$$\mathbf{K} = \mathcal{F}/\sim . \tag{II.13}$$

When $\mathcal{F}$ is choosen as small as possible, it is called *Fundamental Polygon*. We take

$$\mathcal{F} := \left\{ z \in \mathbf{D};\;\; \forall i = 0, ..., 3,\; |g_i(z)| \geq |z|,\;\; |g_i^{-1}(z)| \geq |z| \right\}. \tag{II.14}$$

We can see that $\mathcal{F}$ is a closed regular hyperbolic octogon, of which the boundary $\partial\mathcal{F}$ is the union of eight arcs of circle, that are parts of geodesics of $\mathbf{D}$. We denote P_j, $j \in \mathbb{Z}_8$, the tops of $\mathcal{F}$, and $P_j P_{j+1}$ the eight wedges. The action of Γ on the boundary is described by :

$$\begin{array}{lllllll} P_1 &=& g_3(P_6), & \quad & P_5 &=& g_3^{-1}(P_2), \\ P_2 &=& g_2(P_7), & & P_6 &=& g_2^{-1}(P_3), \\ P_3 &=& g_1(P_8), & & P_7 &=& g_1^{-1}(P_4), \\ P_4 &=& g_0(P_1), & & P_8 &=& g_0^{-1}(P_5), \end{array} \tag{II.15}$$

and

$$\begin{array}{lllllll} P_1P_2 &=& g_3(P_6P_5), & \quad & P_2P_3 &=& g_2(P_7P_6), \\ P_3P_4 &=& g_1(P_8P_7), & & P_4P_5 &=& g_0(P_1P_8). \end{array} \tag{II.16}$$

We define the relation $\sim$ by specifying the classes of equivalence $\dot{z}$ of any $z \in \mathcal{F}$, $\dot{z} := \Gamma(\{z\}) \cap \mathcal{F}$, i.e.

$$z \in \overset{\circ}{\mathcal{F}} \Rightarrow \dot{z} = \{z\}, \tag{II.17}$$

$$\dot{P_j} = \{P_1,\, P_2,\, P_3,\, P_4,\, P_5,\, P_6,\, P_7,\, P_8\}, \tag{II.18}$$

$$z \in \partial\mathcal{F} \setminus \dot{P_j} \Rightarrow \dot{z} = \{z, z_{equiv}\}, \tag{II.19}$$

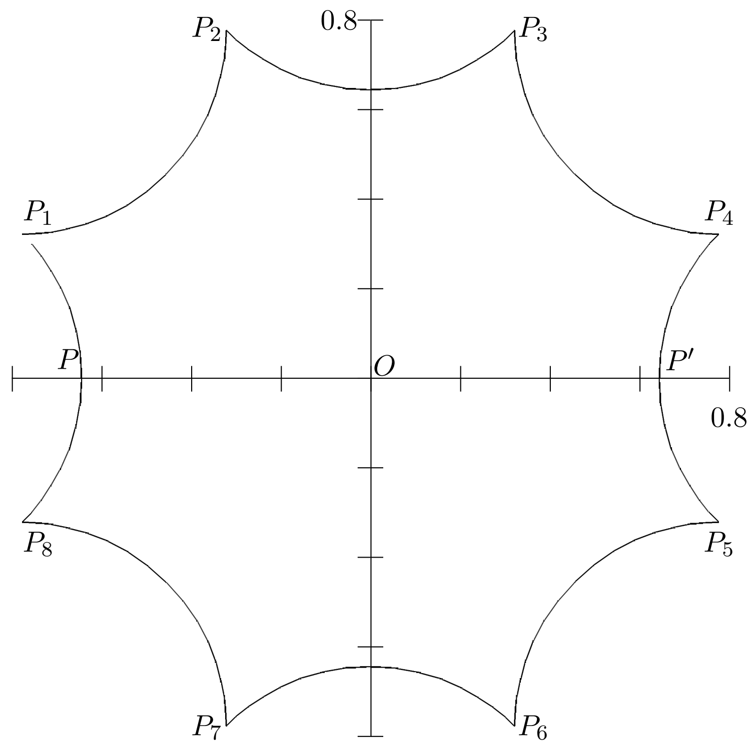

Figure 1: The Fundamental Domain $\mathcal{F}$.

where according (II.16)

$$z \in P_iP_j, \quad P_kP_l = g_a^{\pm 1}(P_iP_j) \Rightarrow z_{equiv} = g_a^{\pm 1}(z) \in P_kP_l. \tag{II.20}$$

We give some metric relations. We denote $P = P_1P_8 \cap \mathbb{R}^-$, $P' = g_0(P) = T_{\tau_1}(P)$. We have

$$d_H(P, P') = 2d_H(O, P') = \tau_1,$$

and so

$$P_1P_8 \subset \left\{ z;\ \left(x + \sqrt{\frac{1+\sqrt{2}}{2}} \right)^2 + y^2 = \frac{\sqrt{2}-1}{2} \right\}, \tag{II.21}$$

and by using the rotations we also have :

$$d_H(P_i, P_{i+1}) = \tau_1 = 2d_H(P, P_1) = 2d_H(O, P),$$

$$d_H(O, P_i) = \tau_2 \quad \text{with} \quad \tanh\frac{\tau_2}{2} = 2^{-\frac{1}{4}}, \ \ d(O, P_i) = 2^{-\frac{1}{4}}.$$

Finally the area of the fundamental domain is $\mu_H(\mathcal{F}) = 4\pi$.

III. The Waves on the Doughnut.

The Laplace Beltrami operator associated with a metric g, is defined by

$$\frac{1}{\sqrt{|g|}}\partial_\mu g^{\mu\nu}\sqrt{|g|}\partial_\nu,\ \ g^{-1}=(g^{\mu\nu})\ ,\qquad |g|=|\det g_{\mu\nu}|.$$

We consider the lorentzian manifold $\mathbb{R}_t\times\mathbf{K}$ endowed with the metric

(III.1) $$g_{\mu\nu}dx^\mu dx^\nu=dt^2-ds^2_{\mathbf{K}}.$$

We study the covariant wave equation

(III.2) $$\partial_t^2\psi-\Delta_{\mathbf{K}}\psi=0,$$

and more generally, the damped wave equation

(III.3) $$\partial_t^2\psi-\Delta_{\mathbf{K}}\psi+a\partial_t\psi=0,$$

whith $0\leq a\in L^\infty(\mathbf{K})$. Here $\Delta_{\mathbf{K}}$ is the Laplace Beltrami operator on $\mathbf{K}$. Since $\mathbf{K}$ is a smooth compact manifold without boundary, $\mathbf{K}$ endowed with its natural domain $\{u\in L^2(\mathbf{K});\ \ \Delta_{\mathbf{K}}u\in L^2(\mathbf{K})\}$ is self-adjoint and the global Cauchy problem is well posed in the framework of the finite energy spaces. Given $m\in\mathbb{N}$, we introduce the Sobolev space

(III.4) $$H^m(\mathbf{K}):=\left\{u\in L^2(\mathbf{K}),\ \nabla_H^\alpha u\in L^2(\mathbf{K}),\ \mid\alpha\mid\leq m\right\}$$

where ∇_H are the covariant derivatives. We can also interpret this space as the set of the distributions $u\in H^m_{loc}(\mathbf{D})$ such that $u\circ g=u$ for any $g\in\Gamma$. Then for all $\psi_0\in H^1(\mathbf{K})$, $\psi_1\in L^2(\mathbf{K})$, there exists a unique $\psi\in C^0\left(\mathbb{R}_t^+;H^1(\mathbf{K})\right)\cap C^1\left(\mathbb{R}_t^+;L^2(\mathbf{K})\right)$ solution of (III.3) satisfying

(III.5) $$\psi(t=0)=\psi_0,\ \ \partial_t\psi(t=0)=\psi_1,$$

and we have

(III.6) $$\int_{\mathbf{K}}\mid\partial_t\psi(t)\mid^2+\mid\nabla_H\psi(t)\mid^2d\mu_H+\int_0^t\int_{\mathbf{K}}a\mid\partial_t\psi(t)\mid^2d\mu_Hdt=Cst.$$

To perform the numerical computation of this solution, we take the fundamental polygon as the domain of calculus. Then the Cauchy problem on $\mathbb{R}_t\times\mathbf{K}$ is equivalent to the mixed problem

(III.7) $$\partial_{tt}\psi-\frac{(1-x^2-y^2)^2}{4}\Big[\partial_{xx}\psi+\partial_{yy}\psi\Big]+a(x,y)\partial_t\psi=0,\ \ (t,x,y)\in\mathbb{R}^+\times\mathcal{F},$$

with the boundary conditions

(III.8) $$\forall(t,z)\in\mathbb{R}\times\partial\mathcal{F},\ \ z\sim z'\Rightarrow\psi(t,z)=\psi(t,z').$$

We denote $H^m(\mathcal{F})=\{u\in L^2(\mathcal{F}),\ \ \forall\alpha\in\mathbb{N}^2,\ \ \mid\alpha\mid\leq m,\ \ \partial^\alpha_{x,y}u\in L^2(\mathcal{F})\}$ the usual Sobolev space H^m for the euclidean metric, and we introduce the spaces $W^m(\mathcal{F})$ that correspond to the spaces $H^m(\mathbf{K})$:

(III.9) $$W^m(\mathcal{F}):=\{u|_{\mathcal{F}};\ \ u\in H^m_{loc}(\mathbf{D}),\ \ \forall g\in\Gamma,\ \ u\circ g=u\}\,,$$

endowed with the norm

(III.10) $$\|u\|_{W^m}:=\|u|_{\mathcal{F}}\|_{H^m(\mathcal{F})}.$$

In particular, we have

(III.11) $$W^1(\mathcal{F})=\left\{u\in H^1(\mathcal{F}),\ \ z\sim z'\Rightarrow u(z)=u(z')\right\},$$

and for all $\psi_0 \in W^1(\mathcal{F})$, $\psi_1 \in L^2(\mathcal{F})$, there exists a unique $\psi \in C^0\left(\mathbb{R}_t^+; W^1(\mathcal{F})\right) \cap C^1\left(\mathbb{R}_t^+; L^2(\mathcal{F})\right)$ solution of (III.5), (III.7) and (III.8), and we have the energy estimate :

$$\text{(III.12)} \quad \begin{aligned} &\int_{\mathcal{F}} \frac{4}{(1-x^2-y^2)^2} \mid \partial_t \psi(t,x,y) \mid^2 + \mid \partial_x \psi(t,x,y) \mid^2 + \mid \partial_y \psi(t,x,y) \mid^2 \, dxdy \\ &\qquad + \int_0^t \int_{\mathcal{F}} \frac{4a(x,y)}{(1-x^2-y^2)^2} \mid \partial_t \psi(t,x,y) \mid^2 \, dxdy = Cst. \end{aligned}$$

Since $a \in L^\infty(\mathcal{F})$ we have a result of regularity : when $\psi_0 \in W^2(\mathcal{F})$, $\psi_1 \in W^1(\mathcal{F})$, then $\psi \in C^0\left(\mathbb{R}_t; W^2(\mathcal{F})\right) \cap C^1\left(\mathbb{R}_t; W^1(\mathcal{F})\right) \cap C^1\left(\mathbb{R}_t; L^2(\mathcal{F})\right)$. In this case the mixed problem can be expressed as a variational problem : ψ is solution iff for all $\phi \in W^1(\mathcal{F})$, we have :

$$\begin{aligned} \frac{d^2}{dt^2} \int_{\mathcal{F}} \frac{4}{(1-x^2-y^2)^2} \psi(t,z)\phi(z) dxdy + \frac{d}{dt} \int_{\mathcal{F}} \frac{4a(z)}{(1-x^2-y^2)^2} \psi(t,z)\phi(z) dxdy \\ - \int_{\mathcal{F}} \Delta_{x,y} \psi(t,z)\phi(z) dxdy = 0. \end{aligned}$$

To invoke the Green formula, we denote $\nu(z)$ the unit outgoing normal at $z \in \partial\mathcal{F}$. We suppose that $P_k P_l = g_a\left(P_i P_j\right)$. Then for $z \in P_i P_j$

$$g_a(\nu_z) = -\nu_{g_a(z)},$$

Since $u \circ g_a = u$, we have for $u \in W^2(\mathcal{F})$

$$\partial_{\nu(z)} u(z) = g_a[\nu(z)].\nabla u(g_a(z)) = -\partial_{\nu(g_a(z))} u(g_a(z)).$$

We deduce that for $u \in W^2(\mathcal{F})$, $v \in W^1(\mathcal{F})$, we have

$$\int_{P_i P_j} v(z) \partial_{\nu(z)} u d\lambda(z) = - \int_{P_k P_l} v(z) \partial_{\nu(z)} u d\lambda(z),$$

and therefore

$$\int_{\partial\mathcal{F}} v(z) \partial_{\nu(z)} u d\lambda(z) = 0, \quad \int_{\mathcal{F}} \Delta_{x,y} u(z) v(z) dxdy = - \int_{\mathcal{F}} \partial_x u \partial_x v + \partial_y u \partial_y v dxdy.$$

We have proved the following

Theorem III.1. *Given $\psi_0 \in W^2(\mathcal{F})$, $\psi_1 \in W^1(\mathcal{F})$, the solution ψ of the Cauchy problem (III.3), (III.5), is the unique solution satisfying (III.5) of the variational problem*

(III.13)

$$\begin{aligned} \forall \phi \in W^1(\mathcal{F}), \quad \frac{d^2}{dt^2} \int_{\mathcal{F}} \frac{4}{(1- \mid z \mid^2)^2} \psi(t,z)\phi(z) dxdy + \frac{d}{dt} \int_{\mathcal{F}} \frac{4a(z)}{(1- \mid z \mid^2)^2} \psi(t,z)\phi(z) dxdy \\ + \int_{\mathcal{F}} \partial_x \psi(t,z) \partial_x \phi(z) + \partial_y \psi(t,z) \partial_y \phi(z) dxdy = 0. \end{aligned}$$

We solve this variational problem by the usual way. We take a family V_h, $0 < h \leq h_0$, of finite dimensional vector subspaces of $W^1(\mathcal{F})$. We assume that

$$\text{(III.14)} \qquad \overline{\cup_{0<h\leq h_0} V_h} = W^1(\mathcal{F}).$$

We choose sequences $\psi_{0,h},\ \psi_{1,h} \in V_h$ such

$$\psi_{0,h} \to \psi_0 \ \ in \ \ W^1(\mathcal{F}), \ \psi_{1,h} \to \psi_1 \ \ in \ \ L^2(\mathcal{F}).$$

We consider the solution $\psi_h \in C^\infty(\mathbb{R}_t; V_h)$ of

(III.15)

$$\forall\phi_h\in V_h,\ \ \frac{d^2}{dt^2}\int_{\mathcal{F}}\frac{4}{(1-\mid z\mid^2)^2}\psi_h(t,z)\phi_h(z)dxdy+\frac{d}{dt}\int_{\mathcal{F}}\frac{4a(z)}{(1-\mid z\mid^2)^2}\psi_h(t,z)\phi_h(z)dxdy$$
$$+\int_{\mathcal{F}}\partial_x\psi_h(t,z)\partial_x\phi_h(z)+\partial_y\psi_h(t,z)\partial_y\phi_h(z)dxdy=0,$$

satisfying $\psi_h(0,.)=\psi_{0,h}(.)$, $\partial_t\psi_h(0,.)=\psi_{1,h}(.)$. Thanks to the conservation of the energy, this scheme is stable :

$$\forall T>0,\ \ \sup_{0<h\leq h_0}\sup_{0\leq t\leq T}\|\psi_h(t)\|_{W^1}+\|\frac{d}{dt}\psi_h(t)\|_{L^2}<\infty.$$

Moreover, when $\psi\in C^2\left(\mathbb{R}^+_t;W^1(\mathcal{F})\right)$, it is also converging :

$$\forall T>0,\ \ \sup_{0\leq t\leq T}\|\psi_h(t)-\psi(t)\|_{W^1}+\|\frac{d}{dt}\psi_h(t)-\frac{d}{dt}\psi(t)\|_{L^2}\to 0,\ \ h\to 0.$$

If we take a basis $\left(e_j^h\right)_{1\leq j\leq N_h}$ of V_h, we expand ψ_h on this basis :

$$\psi_h(t)=\sum_{j=1}^{N_h}\psi_j^h(t)e_j^h,$$

and we introduce

$$\mathbb{M}=\left(M_{ij}\right)_{1\leq i,j\leq N_h},\ \ M_{ij}:=\int_{\mathcal{F}}\frac{4}{(1-\mid z\mid^2)^2}e_i^h(z)e_j^h(z)dxdy,$$
$$\mathbb{D}=\left(D_{ij}\right)_{1\leq i,j\leq N_h},\ \ D_{ij}:=\int_{\mathcal{F}}\frac{4a(z)}{(1-\mid z\mid^2)^2}e_i^h(z)e_j^h(z)dxdy,$$
$$\mathbb{K}=\left(K_{ij}\right)_{1\leq i,j\leq N_h},\ \ K_{ij}:=\int_{\mathcal{F}}\partial_x e_i^h(z)\partial_x e_j^h(z)+\partial_y e_i^h(z)\partial_y e_j^h(z)dxdy,$$
$$X:=\begin{pmatrix}\psi_1^h\\ \psi_2^h\\ .\\ .\\ .\\ \psi_{N_h}^h\end{pmatrix}.$$

Then the variational formulation is equivalent to

(III.16) $$\mathbb{M}X''+\mathbb{D}X'+\mathbb{K}X=0.$$

This differential system is solved very simply by iteration by solving

(III.17) $$\mathbb{M}(X^{n+1}-2X^n+X^{n-1})+\frac{\Delta T}{2}\mathbb{D}(X^{n+1}-X^{n-1})+(\Delta T)^2\mathbb{K}X^n=0.$$

We know that this scheme is stable, and so convergent by the Lax theorem, when

(III.18) $$\sup_{X\neq 0}\frac{<\mathbb{K}X,X>}{<\mathbb{M}X,X>}<\frac{4}{\Delta T^2},$$

and if there exists $K>0$ such that

(III.19) $$\forall h\in]0,h_0],\ \ \forall\phi_h\in V_h,\ \ \|\nabla_{x,y}\phi_h\|_{L^2(\mathcal{F})}\leq\frac{K}{h}\left\|\frac{2}{1-\mid z\mid^2}\phi_h\right\|_{L^2(\mathcal{F})},$$

the CFL condition

(III.20) $$K\Delta T<2h,$$

is sufficient to assure the stability and the convergence of our scheme.

IV. Numerical resolution

IV.1. **Mesh.** First of all we construct the boundary $\partial\mathcal{F}$ from the equation (II.21) and we perform a discretization that is equidistant for the hyperbolic metric. Next we use the mesh generators Emc2$^{\text{TM}}$and bamg$^{\text{TM}}$created by INRIA. If we only use Emc2$^{\text{TM}}$, the mesh contains too many vertices and is not suitable for the hyperbolic metric. So a first mesh is created by Emc2$^{\text{TM}}$. We also consider a circle which is uniformly discretized with the same hyperbolic step than the exterior geometry. The radius is choosen as the final mesh is almost uniform. At last, we impose on every point of the exterior and interior geometry a metric, in the sense of bamg$^{\text{TM}}$. This software can next create a mesh which is more uniform, with respect to the hyperbolic metric, than the first mesh, and that has a reasonnable number of vertices.

To test the uniformity of the mesh, we compute the extrema of the hyperbolic distance between two neighbor vertices. As a check of the accuracy of the meshes we evaluated the area of the polygons created by the meshes, and we compared to 4π (area of the domain). Here are some examples:

$Mesh$	$number\ of\ vertices$	$\max d_H$	$\min d_H$	$area/4\pi$
$Mesh1$:	7448	0.087	0.027	1.00012
$Mesh2$:	17574	0.049	0.0177	1.00007
$Mesh3$:	37329	0.036	0.012	1.00003
$Mesh4$:	67517	0.027	0.009	1.000018

In our meshes, the greater hyperbolic distance between consecutive vertices is not reached near the exterior boundary. To give an idea of the accuracy of this dicretization, we show in the following figure, a very rough mesh :

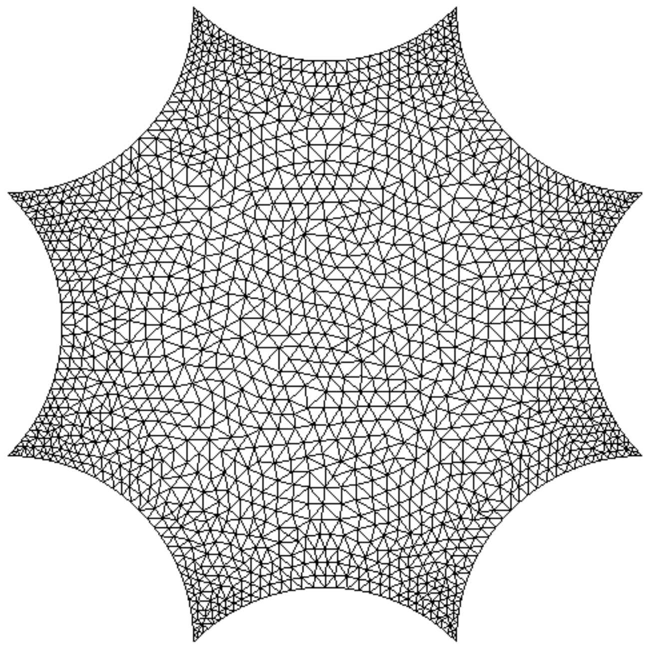

Figure 2: A very rough mesh with 1756 vertices.

IV.2. V_h **space.** We construct the finite element spaces of $\mathbb{P}_1$ type. We note $\mathcal{T}_h$ all triangles of a mesh, and $\mathcal{F}_h = \cup_{K\in\mathcal{T}_h} K$.

$$V_h := \left\{ v : \mathcal{F}_h \to \mathbb{R},\ v \in \mathcal{C}^0(\mathcal{F}_h),\ \forall K \in \mathcal{T}_h\ v_{|K} \in \mathbb{P}_1(K),\ M \sim M' \Rightarrow v(M) = v(M') \right\}$$

If M_i and M_j denote two vertices of the mesh, we define a basis $\left(e^h_j\right)_{1\leq j\leq N_h}$ of V_h by:

(1) If $M_j \notin \partial\mathcal{F}$: $e^h_j(M_i) = \delta_{ij}$

(2) If M_j is a P_j point: $e^h_j(M_i) = \begin{cases} 1 & if\ M_i = P_i \\ 0 & otherwise \end{cases}$

(3) If $M_j \in \partial\mathcal{F}$, and is not a P_j point: $e^h_j(M_i) = \begin{cases} 1 & if\ M_i \sim M_j \\ 0 & otherwise \end{cases}$

In particular, we have to determine the equivalent points on $\partial\mathcal{F}$. To that, we write a program implementing the relations (II.20).

The number of nodes N_h is the sum of the number of the vertices which are not in $\partial\mathcal{F}$, the number of vertices which are on four consecutive arcs of $\partial\mathcal{F}$ without beeing a P_i point, and one (because all P_i points are equivalent to one of them).

IV.3. **Matrix form of the problem.** $\mathbb{K}(i,j)$ and $\mathbb{M}(i,j)$ are found with a numerical integration using the value at the middle of the edges of the triangles.

The stiffness matrix $\mathbb{K}$ and the mass matrix $\mathbb{M}$ are sparse and symetric matrices. So we choose a Morse stockage of their lower part, and all of the calculations will be performed with this stockage.

To solve the linear problem we use a preconditionned conjugate gradient method. The preconditionner is an incomplete Choleski factorisation, and the starting point is the solution obtained with a diagonal preconditionner.

IV.4. **Initial data.** We consider the case where the initial velocity $\psi_1 = 0$, i.e. $X^0 = X^1 = 0$, and we choose first initial datas with a more or less small support near a given point. For instance, for the wave depicted in Figure 3, we have taken

$$\psi_0(x,y) = 100e^{\frac{1}{100x^2+100y^2-1}}, \quad for\ x^2+y^2 < \frac{1}{100},\ \psi_0(x,y) = 0, \quad for\ x^2+y^2 \geq \frac{1}{100}. \tag{IV.1}$$

IV.5. **Discretized energy.** In order to see the stability of our method we perform $E_d(t)$ the discretized energy at the time t:

$$E(n\Delta t) = \left\langle \mathbb{M}\ \frac{X^n - X^{n-1}}{\Delta t}\ ,\ \frac{X^n - X^{n-1}}{\Delta t} \right\rangle + \left\langle \mathbb{K}\ X^{n-1}\ ,\ X^n \right\rangle$$

It is well known that our schema is conservative when $a = 0$, hence E_d must be invariant all along the resolution. We test this property with the previous initial data.

	$E_d(0)$:	$E_d(100)$:	
		with Δt = 0.001 :	with Δt = 0.0005 :
$Mesh1$:	8455.89602005935	8455.89602005865	
$Mesh2$:	8484.17400988788	8484.17400988815	8484.17400988936
$Mesh3$:	8494.43409419468	8494.43409419464	8494.43409419511
$Mesh4$:	8498.39023175937	8498.39023175945	8498.39023175923

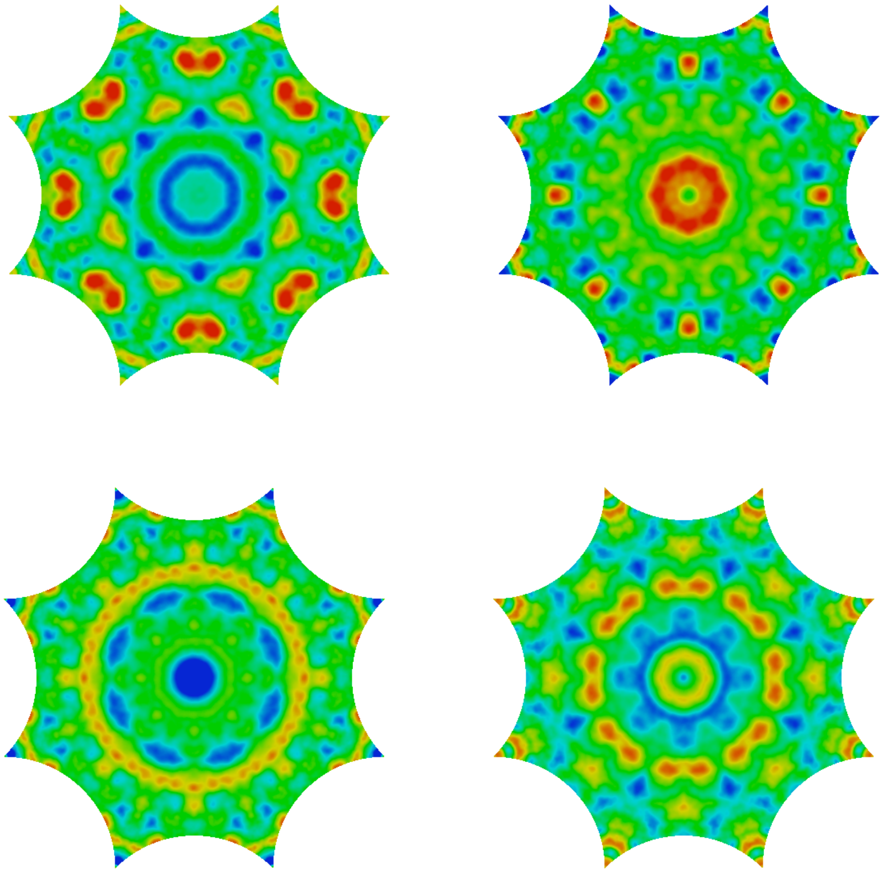

Figure 3: The transient wave at $t = 34$, $t = 35$, $t = 36$, $t = 37$.

IV.6. **Eigenvalues.** We test our scheme in the time domain, by looking for the eigenvalues of the hamiltonian when $a = 0$. Since the Laplace-Beltrami operator $\Delta_{\mathbf{K}}$ on the Hyperbolic Double Doughnut is a non positive, self-adjoint elliptic operator on a compact manifold, its spectrum is a discrete set of eigenvalues $-q^2 \leq 0$, and by the Hilbert-Schmidt theorem, there exists an orthonormal basis in $L^2(\mathbf{K})$, formed of eigenfunctions $(\psi_q)_q \subset H^\infty(\mathbf{K})$ associated to q^2, i.e.

$$-\frac{(1-x^2-y^2)^2}{4}\Big[\partial_{xx}\psi_q + \partial_{yy}\psi_q\Big] = q^2\psi_q, \quad \psi_q \in W^\infty(\mathcal{F}). \tag{IV.2}$$

We take $\psi_0 = \frac{1}{2\sqrt{\pi}}$. Therefore any finite energy solution $\psi(t,x,y)$ of $\partial_t^2\psi - \Delta_{\mathbf{K}}\psi = 0$ has an expansion of the form $\sum_q e^{iqt}\psi_q(x,y)$ (such expansions exist also for the damped wave equation, when $a > 0$, see [6]). More precisely, if we denote $<,>$ the scalar product in $L^2(\mathbf{K})$, we write

$$\begin{aligned}\psi(t,x,y) =& \frac{1}{4\pi}(< \partial_t\psi(0,.), 1 > t + < \psi(0,.), 1 >)\\ &+ \sum_{q\neq 0} < \partial_t\psi(0,.), \psi_q > \frac{\sin qt}{q}\ \psi_q(x,y) + \ < \psi(0,.), \psi_q > \cos qt\ \psi_q(x,y).\end{aligned} \tag{IV.3}$$

To compute the eigenvalues q, we investigate the Fourier transform in time of the signal $\psi(t,x,y)$ in the case where $\partial_t\psi(0,x,y)=0$. We fix some large $T>>1$, and we put $\Psi_\omega(X):=\int_0^T\psi(t,X)e^{i\omega t}\,dt$. Then $\Psi_\omega(t)\sim CT^2(\omega^2-q^2)^{-1}$, $T\to\infty$. Practically, during the time resolution of the equation we store the values of the solution at some points M, including the origin, P', M_0 near P_4, for the discrete time $k\Delta t$, $N_i\leq k\leq N_f$. We choose the initial step N_i in order to the transient wave is stabilized, that to say $N_i\Delta t$ is greater than the diameter of the doughnut, i.e. $N_i\Delta t\geq 2*d_H(0,P_i)\simeq 4.8969$. Then we compute a DFT of $(\psi_h(k\Delta t,M))_{N_i\leq k\leq N_f}$ with the free FFT library fftw. Let us note $(\Psi_j(M))_{0,N_f-N_i+1}$ the result. If $\Psi_j(M):=\sum_{k=0}^{N_f-N_i+1}\psi((N_i+k)\Delta t,x,y)e^{-(\frac{2\pi}{N_f-N_i+1}jk)i}$, we search the values $jmax_1$, $jmax_2$, ... for which $(\|\Psi_j(M)\|^2)_j$ has a maximum. Then the eigenvalues found by the algoritm expressed as:

$$q=\frac{2\pi}{(N_f-N_i+1)\Delta t}jmax$$

We have made a lot of tests by varying parameters such as: Δt, N_i, N_f, the mesh, the observation point M. With the initial data (IV.1), we find the following values for q :

$$\begin{array}{ll} 1,96\pm 0,02 & 2,85\pm 0,04\\ 4,34\pm 0,06 & 4,83\pm 0,06\\ 6,00\pm 0,06 & 6,63\pm 0,06 \end{array}$$

These results agree with the results obtained with a stationnary method with a mesh of 3518 vertices in [4].
Alternatively, we could also use the power spectrum and calculate the square of the modulus of $\Psi_\omega(x,y)$

$$\begin{aligned}\|\Psi_\omega\|^2_{L^2(\mathbf{K})}=&\ \frac{1}{4\pi}\,|<\psi(0,.),1>|^2\\ &+\frac{1}{2}\sum_q<\psi(0,.),\psi_q>^2\left[\frac{\sin^2\frac{(q+\omega)T}{2}}{(q+\omega)^2}+\frac{\sin^2\frac{(\omega-q)T}{2}}{(\omega-q)^2}+\frac{1}{\omega^2-q^2}\left(\cos^2\omega T-\cos\omega T\cos qT\right)\right]\end{aligned}$$

Therefore, for an eigenvalue q_0:

$$\|\Psi_\omega\|^2_{L^2(\mathbf{K})}\geq<\psi(0,.),\psi_q>^2\left(\frac{T^2}{4}-\frac{T}{2q}\right).$$

IV.7. **Damped waves.** We test our scheme for the damped wave equation (III.3) when the damping function $a\geq 0$ is non zero (for deep theoretical results, see [6], [8], [9]). We know that the energy of any finite energy solution decays exponentially (uniformly with respect to the initial energy) iff the dumping a satisfies the assumption of geometric control introduced by J. Rauch and M. Taylor in [9]. This condition means

$$\int_0^\infty a(x(t),y(t))dt=+\infty \tag{IV.4}$$

for any geodesic $(x(t),y(t))$. Since the geodesic flow on the compact Riemaniann manifold with constant negative curvature is very chaotic (more precisely ergodic, mixing, Anosov, Bernouillian see e.g. [1], [2], [5]), it is sufficient to have $a>0$ near $\partial\mathcal{F}$. Nevertheless we constat an exponential decay for some solution, even if we choose a dumping function a equal to a positive constant on very small support that does not satisfy (IV.4) : $a>0$ only on one triangle and its close neighbors.

The next figures are obtained with mesh3 and a defined by:

$$a(x,y)=0,\ \text{for}\ ,\mid z\mid<0.6\quad;\quad a(x,y)=0.1,\ \text{otherwise}.$$

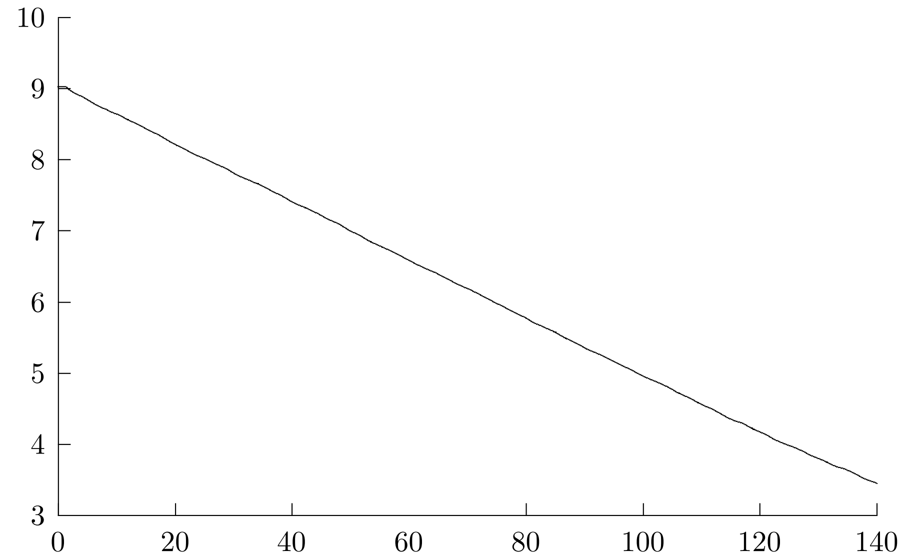

Figure 4: Logarithm of the energy as a function of time.

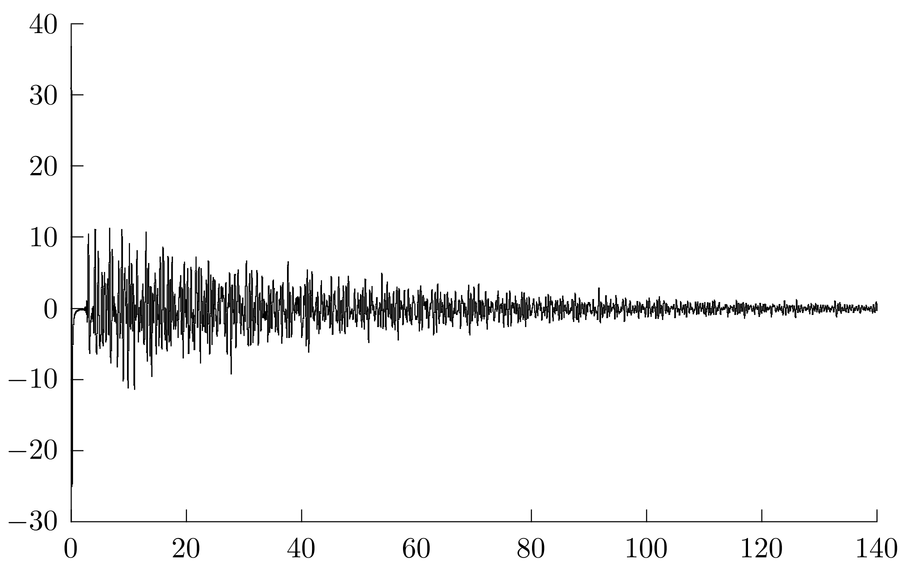

Figure 5: Solution at the origin.

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